# Shocking Stories[1]

S. Levtchenkova and A.R. Pagan

Australian National University

J.C. Robertson

Federal Reserve Bank of Atlanta and Australian National Univeristy

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[1]Address correspondence to: Professor Adrian Pagan, Research School of Social Sciences, The Australian National University, Canberra, ACT 0200, AUSTRALIA (E-mail: arpagan@coombs.anu.edu.au). Our thanks for comments on earlier versions of this paper go to Trevor Breusch. We did not use his suggested title of "The Shocking Truth", although it was certainly tempting to do so. We are also grateful to Warwick McKibbin for performing many simulations on the MSG2 model for us. 

**Abstract**

The paper provides a survey of methods that decompose multivariate series into permanent and transitory components by using ideas drawn from the co-integration literature. We adopt a two stage procedure to effect the decomposition. In the first stage a basic set of permanent and transitory components is formed by using standard definitions of the shocks which they are constituted from. The resulting measurements are not unique and further information needs to be employed to get uniqueness. Such information can come in many forms but a particularly important one involves the values of the long-run multipliers for permanent shocks that are available from many calibrated models. A comparison of the methods of effecting the decomposition is performed using a well known data set.

**Keywords**: permanent shock; transitory shock; co-integration; calibrated model; structural VAR.

# 1 Introduction

Many see quantitative economic research as the construction of stories that rationalize observations. The plausibility of the recounted stories frequently depends upon their coherence with some theoretical framework. When the "cointegration revolution" began it was often said that one of its attractions was its ability to provide a way of incorporating ideas about the "long-run", gained from theoretical models, into quantitative models. That contention has manifested itself primarily by the examination of the implications of long-run behaviour for the cointegrating vectors. But such a focus seems to be too narrow. Theoretical models provide more information than this, as can be seen from studies which feature "long-run" relations in systems composed of variables that are not even cointegrated, e.g. Lastrapes and Selgin (1995). This less studied branch of the literature emphasizes the effects of permanent and transitory shocks upon variables rather than cointegration *per se*. One advantage of this different perspective is that a good deal of modern macroeconomics involves studying such shocks. Results concerning the impact of shocks are many and varied, ranging from general statements such as "supply shocks have a permanent effect upon output whereas demand shocks do not" to the specific numerical values assigned by "calibrated" models. The latter are a rich source of information, encompassing results from both stochastic general equilibrium models, such as those in the Real Business Cycle class, and other models with a strong theoretical base that are frequently used in policy analyses. One example of a calibrated model, to which we shall frequently refer, is the MSG2 model of the global economy set out in McKibbin and Sachs (1991) and McKibbin (1997).

The presence of a common language might lead one to expect a close connection between the cointegration and shock analysis literatures. But, in fact, the relationship is a little hazy. Part of the problem lies in the fact that there are many papers in the econometric literature that use the cointegration properties of the data to provide a decomposition into permanent and transitory components with seemingly no reference to any economic story at all, e.g. Gonzalo and Granger (1995), and certainly not one associated with any theoretical model. Moreover, there are also "reverse engineering" papers in which a story is told but the theoretical model is designed to be compatible with the cointegration properties of the data. Examples of the latter would be King, Plosser, Stock and Watson (1991) (KPSW) and the papers inspired by it, such as Mellander, Vredin and Warne (1992). Because of this loose connection, there seems a good case for a paper that places the literature into a framework that enables one to see more clearly how the stories (theory) enter into the picture, and what stories are being told with each solution. To do this we focus upon the way in which information is brought to bear upon methods of estimating the impulse responses with respect to permanent and transitory shocks. This information is of a diverse kind and can be used in many ways. In a decomposition such as that by Gonzalo and Granger it is very general, while in the KPSW variant it is much more specific. Moreover, there is a tendency for it to become increasingly specific as one gets closer to replicating any chosen theoretical model. As we will make clear there is a hierachy of steps that can be followed when imposing the available information. On the top level it is possible to provide a split into temporary and permanent components that uses the minimal amount of information to discriminate between the two. Extra information can then be used to refine this division by combining the "basic" set of permanent and transitory shocks into a new set which is consistent with the expanded information. As the information becomes more specific the stories become more recognizable.

In the following section we begin by looking at the case when all shocks are transitory. It may seem odd to do this but the analysis is formally the same as when all shocks are permanent, the situation considered in the first part of section 3. Things become more complex when both permanent and transitory shocks are allowed to co-exist and a study of the issues raised by the joint presence of both types of shocks constitutes the remaining parts of the section. Section 4 completes the analysis by discussing systems that have a combination of integrated and non-integrated variables. Section 5 gives a comparison of the different methods of isolating permanent and transitory components using a data set from KPSW, and section 6 concludes.

## 2 Stationary Systems

Let $y_t$ be an $n \times 1$ vector of $I(0)$ variables. Two tasks need to be performed. The first is to provide a convenient way to *summarize* the data while the second is to *interpret* it in an informative way. Regarding the first we will assume that the second moment properties of the data are summarized by a $p$'th order VAR,

$$A(L)y_t = e_t, \tag{1}$$

where $A(L) = I_n - A_1 L - ... - A_p L^p$ and $e_t \sim i.d.(0, \Omega)$.[1] For the second issue one notes that modern macroeconomics tends to focus upon the effects of "shocks" with names given to them such as "money demand", "supply", "real", etc. It is in ascribing names to the shocks that one constructs a story about them. Once one has defined the nature of the shocks in some way it is possible to interpret the data.

There are two general approaches that lead to a naming of shocks. In the first, the shocks are regarded as being the disturbances terms in $n$ structural equations that are given names such as "money supply function", "aggregate supply function" etc. Then it is differences in the specification of these equations which defines the nature of the shocks. These relations may be written as

$$B(L)y_t = \varepsilon_t,$$

where $B(L) = B_0 - B_1 L - ... - B_p L^p$, and $B_0 e_t = \varepsilon_t \sim i.d.(0, \Sigma)$. Although the shocks are differentiated by $\{B_j\}_{j=0}^{p}$ and $\Sigma$ *in toto*, attention typically focuses upon the nature of $B_0$ and $\Sigma$. Some of the elements of these matrices are fixed through normalizations and restrictions and some will need to be estimated. In doing the latter a constraint that will be enforced is that the chosen structural representation exactly reproduces the statistical characteristics of the data summarized by the estimated parameters of (1), $\left\{\tilde{A}_j\right\}_{j=1}^{p}$ and $\tilde{\Omega}$. The question which then arises is whether it is possible to uniquely determine $\{B_j\}_{j=0}^{p}$ and $\Sigma$ from the implied relationships

$$B_0 A_j = B_j, j = 1, ..., p \quad (2)$$

$$B_0 \Omega B_0' = \Sigma, \quad (3)$$

where $A_j$ and $\Omega$ are fixed at their estimates $\tilde{A}_j$ and $\tilde{\Omega}$.[2] It is clear from counting the number of equations that it is impossible to identify $B_0$ if one maintains the assumption that all elements in $\{B_j\}_{j=0}^{p}$ are unknown, since that means (2) has no information in it. Accordingly, attention has focussed upon (3), with Sims (1980) making $B_0$ triangular and $\Sigma = I_n$. The triangularity of $B_0$ imposes $n(n-1)/2$ exclusion restrictions while the requirement that $\Omega = B_0^{-1}(B_0')^{-1}$ imposes $n(n+1)/2$ non-linear restrictions on $B_0$. Later Sims and others allowed $B_0$ to have patterns of zeros other than the triangular one, provided their number did not exceed that in the triangular form.[3] This latter development is sometimes described as a "structural VAR" approach, but this really a misnomer since it simply provides another way of describing $B_0$, i.e. it tells a different story to that implied by a particular recursive model. Although the interpretation offered in support of restrictions on $B_0$ ostensibly arises from some prior economic theory, in practice most ideas concerning the restrictions on $B_0$ come from past empirical work or introspection. A short hand way of describing the above would be as a *structural VAR methodology* (SVM).

To measure the responses of variables to particular shocks one needs to know the time paths of the $y_t$ as a function of current and past shocks. This is given by the vector MA representation

$$y_t = C(L)\varepsilon_t, \quad (4)$$

where $C(L) = C_0 + C_1 L + C_2 L^2 - ...$ . The representation in (4) gives rise to the second approach in which the information used to characterize the shocks stems primarily from the nature of the multipliers $C_j$. We will refer to this generically as the *structural MA methodology* (SMM). A great variety of information is subsumed under this heading. For example, it is possible that the impact responses of $y_t$ to selected shocks, i.e. elements of $C_0$, are known. Alternatively, functions of $C_j$ such as $\sum_{j=1}^{q} C_j$ may be prescribed. "Calibrated models" are particularly good at providing this latter type of information but are not the only sources of it.[4] For example, a calibrated RBC model would produce impulse responses to an unanticipated shock to technical progress, while a model such as MSG2 details responses to unanticipated money and productivity shocks. Consequently, there is a lot of information produced by calibrated models pertaining to $C(L)$ and the major challenge is to make it tractable enough to be used for the purpose of measuring the impact of shocks from the data.

Regardless of whether one is working within the SVM or SMM, the symbol $\varepsilon_t$ will be used to denote their respective "economic shocks", reflecting the fact that the shocks in both approaches share common names, like "money", and it is only through the process of definition that they differ. In the SMM tradition one still has to have a way of relating the shocks $\varepsilon_t$ to the data via the VAR errors $e_t$, and this is accomplished by taking the VAR errors to be some non-singular transformation of the economic shocks i.e. $T\varepsilon_t = e_t$. With such an assumption the estimation problem becomes one of how to convert the relations $C(L) = A(L)^{-1}T$ and $T\Sigma T' = \Omega$ into a method for determining the weights $T$. In the SVM it is the VAR errors that are combined to define the economic shocks as $B_0 e_t = \varepsilon_t$, and the estimation problem involves converting the relations $B(L) = B_0 A(L)$ and $\Sigma = B_0 \Omega B_0'$ into methods for determining $B_{0.}$

The distinction between the methodologies should not be pushed too far, however, since $B_0^{-1}$ can always be interpreted as $T$ in the SMM, and vice versa. Also, as we will discuss below, it is sometimes easy to convert restrictions upon $C(L)$ into restrictions upon $B(L)$ and this can lead to some useful insights which would not be available if one was not within the SVM. Nevertheless, we believe that the dichotomy does capture general differences in attitudes towards where information comes from and how it is to be used to distinguish between shocks. It certainly seems to be the case that those working within the SVM emphasize the determination of $B_0$ as the way of identifying shocks and do not think primarily in terms of restrictions on $C(L)$.

Because $y_t$ is assumed stationary, the shocks to them will be *transitory;* any such shock will be defined as one whose associated column of $C_\infty$ is zero, where $C_\infty$ shows the "long-run" impact of the shocks. When all shocks are transitory it follows that $C_\infty = 0$, i.e. the impact of a transitory shock upon the level of all the $I(0)$ variables is zero "in the long-run". Clearly, when the variables are stationary, one can not use this outcome to differentiate between shocks. But it may be possible to do so based on some other features of $C(L)$, such as restrictions on the partial sums of the $C_j$ or the "mean lag" of $C(L)$. As a precursor to the discussion of nonstationary systems it will be useful to pay particular attention to $C(1) = \sum_{j=0}^{\infty} C_j$.[5]

Retaining the requirement that the data, as summarized by the fitted VAR, is to be exactly represented, means that the estimate of $T$ (in the SMM) or $B_0^{-1}$ (in the SVM) has to obey

$$C(1) = A(1)^{-1}T = A(1)^{-1}B_0^{-1}. \tag{5}$$

In fact there are many ways of re-expressing (5). Which one of these is most useful depends a good deal upon the type of information available. If rows of $C(1)$ are known then the relation $A(1)^{-1} = C(1)B_0 = C(1)T^{-1}$ produces linear restrictions upon the corresponding rows of $B_0$ (or $T^{-1}$), whereas (5) might be preferred if one knows columns of $C(1)$, which tends to be the information provided by calibrated models. For example, McKibbin et al (1998) give an application in which the columns of $C(1)$ are computed from selected shocks applied to the MSG2 model of the world economy. Versions like (5) are also very useful if information is also available about the contemporaneous impact of shocks, e.g. that money has no immediate impact upon output, since that imposes exclusion restrictions directly on $T$.

In general, linear restrictions imposed upon $C(L)$ translate into non-linear constraints on $B(L) = C(L)^{-1}$, and vice versa. One exception is the case of triangular exclusion restrictions. For example, when $C_0$ is assumed lower triangular $B_0$ is also lower triangular. Similarly, when $C(1)$ is lower triangular so is $B(1) = \sum_{j=0}^{p} B_j$. Consequently, it is not surprising that the triangularity of $C_0$ or $C(1)$ has seen widespread use by those working in both the SVM and SMM. The case when $C_0$ is triangular is fairly straightforward. To explore the impact of triangularity of $C(1)$ in a little more detail let $n = 2$, and partition $B(L)$, $C(L)$ accordingly. From the relation $C(1)B(1) = I_n$ one has a system of four equations

$$\begin{aligned} C_{11}(1)B_{11}(1) + C_{12}(1)B_{21}(1) &= 1 \\ C_{11}(1)B_{12}(1) + C_{12}(1)B_{22}(1) &= 0 \\ C_{21}(1)B_{11}(1) + C_{22}(1)B_{21}(1) &= 0 \\ C_{21}(1)B_{12}(1) + C_{22}(1)B_{22}(1) &= 1. \end{aligned}$$

With $C_{12}(1) = 0$, and as long as $C_{11}(1) \neq 0$, it follows from the second equation that $B_{12}(1) = 0$. Thus, the restriction $C_{12}(1) = 0$ is equivalent to a homogeneity restriction on the first structural relation in $B(L)y_t = \varepsilon_t$. Making $p = 1$ and defining the elements of matrices as $B_0 = \begin{bmatrix} 1 & b_{12}^0 \\ b_{21}^0 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^1 & b_{22}^1 \end{bmatrix}$, we have $B_{12}(1) = b_{12}^0 - b_{12}^1 = 0$. Imposing the latter restriction results in an estimable system of the form

$$\begin{aligned} y_{1t} + b_{12}^0(y_{2t} - y_{2t-1}) &= b_{11}^1 y_{1t-1} + \varepsilon_{1t} \\ y_{2t} + b_{21}^0 y_{1t} &= b_{21}^1 y_{1t-1} + b_{22}^1 y_{2t-1} + \varepsilon_{2t}, \end{aligned}$$

showing that a linear constraint has been imposed upon the coefficients of $B(L)$. In particular, this constraint means that $y_{2t-1}$ can be used as an instrument for $\Delta y_{2t}$ to estimate $b_{12}^0$, and this clearly defines the shock $\varepsilon_{1t}$. The assumption that $\varepsilon_{2t}$ is uncorrelated with $\varepsilon_{1t}$ then frees up $\varepsilon_{1t}$ for use as an instrument in estimating the coefficients in the second equation. This instrumental variables approach to estimation was used in Shapiro and Watson (1988), and its properties are studied in Pagan and Robertson (1998).

The analysis above is also useful for looking at one criticism that has been made in the literature regarding the utility of restrictions based upon $C(1)$. Faust and Leeper (1997) argue that the resulting estimator of $B_0$ (or $T$) is hopelessly imprecise. Suppose that we modify the previous example by adding the assumption that $b_{21}^0 = 0$, and, in order to ensure that the structural model is observationally equivalent to the VAR, $\Sigma$ is no longer required to be diagonal. From the relation $B_0A(1) = B(1)$, and with $B_{12}(1) = 0$, we have

$$\begin{bmatrix} 1 & b_{12}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11}(1) & A_{12}(1) \\ A_{21}(1) & A_{22}(1) \end{bmatrix} = \begin{bmatrix} B_{11}(1) & 0 \\ B_{21}(1) & B_{22}(1) \end{bmatrix}$$

from which it follows that

$$b_{12}^0 = -A_{12}(1)/A_{22}(1).$$

Clearly the distribution of the estimator of $A_{ij}(1)$ is central to the statistical behaviour of the estimator of $b_{12}^0$. If the order of the VAR is finite we would expect the maximum likelihood estimator $\tilde{A}_{22}(1)$ to be consistent and for $N_{12}^{1/2}\tilde{A}_{12}(1)$ to satisfy a central limit theorem, yielding standard asymptotic properties for the implied estimate of $b_{12}^0$. But, if the true order of the VAR is infinite, so that the fitted one is only an approximation, the situation is more complex. The literature on this allows $p \to \infty$ but in such a way that $p^3/N \to 0$ as $N \to \infty$, and also requires that the difference between the approximation and the true infinite order VAR be $0(N^{1/2})$. Under these restrictions the $\tilde{A}_j$ are consistent and $N_{12}^{1/2}\tilde{A}_j$ have a limiting normal distribution, but their sum $\tilde{A}(1) = \sum_{j=1}^{p} \tilde{A}_j$ has a variance that diverges as $N \to \infty$, see Lütkepohl (1990). It is hard to know what to make of the practical relevance of this point other than to say that the specification of the order of the VAR is not innocuous when utilising restrictions on $B(1)$ or $C(1)$. A similar point is made by Cooley and Dwyer (1998).

We can also use this simple bivariate framework to look at another issue that has received some attention in the literature. In particular, is there a connection between the assumption that $C(1)$ is triangular, and the standard approach where

$B_0$ is made triangular? Suppose one modified the previous example by making $B_0$ lower triangular instead of upper triangular. Then the relation $B_0 A(1) = B(1)$ has the form

$$\begin{bmatrix} 1 & 0 \\ b_{21}^0 & 1 \end{bmatrix} \begin{bmatrix} A_{11}(1) & A_{12}(1) \\ A_{21}(1) & A_{22}(1) \end{bmatrix} = \begin{bmatrix} B_{11}(1) & 0 \\ B_{21}(1) & B_{22}(1) \end{bmatrix}.$$

Clearly, this can only hold when $A_{12}(1) = \sum_{i=1}^{p} a_{12}^{i} = 0$, which implies that either only lagged *changes* of $x_{2t}$ enter into the VAR equation for $x_{1t}$, or lags of $x_{2t}$ are excluded entirely. Given this form for $A(1)$, the lower triangularity of $B_0$ and $C(1)$ are not independent restrictions. Consequently, an additional restriction is still required in order to uniquely determine $b_{21}^0$; $\Sigma$ being diagonal would suffice.

# 3 Non-stationary I(1) Systems

When variables are allowed to be $I(1)$ a new distinction between shocks can be made, namely whether they are *permanent* or transitory. Thus one needs to make a division in the analysis according to this feature. Given the way permanent shocks will be defined, a system of $I(1)$ variables will have $n - r$ of them, where $r$ is the number of cointegrating vectors. Consequently, any analysis is naturally done by distinguishing between the situations when there is and is not cointegration. Permanent shocks may be of interest not only for their dynamic impact upon variables, but also because they are the building blocks of the common trends that are taken to drive $I(1)$ variables. If one is interested in this aspect, a trio of questions becomes relevant- how does one *determine* the number of common trends, how does one *find* the common trends and how should one *interpret* the common trends? The first of these relates to the number of co-integrating vectors in the system and, as it has been well covered in many papers, will be ignored here. Answers are provided to the other two.

## 3.1 I(1) Variables are Not Cointegrated

Let $y_t = \Delta x_t$ be such that $x_t$ consists of $n$ $I(1)$ variables that are not cointegrated. In this case the relevant VAR is in terms of the first differences

$$A(L)\Delta x_t = e_t, \tag{6}$$

while the relation between $\Delta x_t$ and the economic shocks is given by

$$\Delta x_t = C(L)\varepsilon_t,$$

and impact of the shocks upon the *levels* of $x_t$ is given by the elements of $\Psi(L) = (1-L)^{-1}C(L)$. A permanent shock will be defined as one whose associated column of $\Psi_\infty$ has rank one. This means that not all the elements of that column are zero. This contrasts with the case of transitory shocks in which the relevant columns of $\Psi_\infty$ will be zero. A key relationship in the analysis to follow is that $C(1) = \Psi_\infty$, and therefore the cumulative response of $\Delta x_t$ to the shocks has the interpretation of showing the long run impact of the same shocks upon the level of the variables $x_t$. Apart from this modification, the analysis proceeds in exactly the same way as in the preceding section, simply by a re-definition of the variables $(y_t)$ from the original levels into differenced ones. Studies by Ahmed, Ickes, Wang and Yoo (1993), Rogers and Wang (1993), Lastrapes and Selgin (1994, 1995), Tallman and Wang (1995), and Bullard and Keating (1995), *inter alia*, all proceed in this way. Of course, this specification relies on an assumption that the series are $I(1)$ and that there is no cointegration. The implied restrictions on the VAR representation of $x_t$ can be tested.

To illustrate the basic ideas when restrictions on $C(1)$ are applied consider the example in Lastrapes and Selgin (1995). This has $n = 4$ variables given by the nominal interest rate $r_t$, the log of the level of output $o_t$, the log of the real money stock $m - p_t$, and the log of the nominal money stock $m_t$. There are four shocks, which are referred to as an IS shock, an aggregate supply shock, a money demand shock and a money supply shock. They adopt what is a very popular specification in this literature viz that $C(1)$ is triangular, with the form

$$C(1)_{LS} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix},$$

where an asterisk indicates that the value in that cell is unknown, but assumed non-zero. Thus, $C(1)_{LS}$ contains $n(n-1)/2$ restrictions, and combined with the assumption that the economic shocks are uncorrelated with $\Sigma = I_n$, one then has enough restrictions to be able to uniquely determine $T$.

This approach can be contrasted with what one might do when complete information about $C(1)$ is available. To illustrate this we simulated the long run response of the four variables above to four permanent shocks in the MSG2 model documented in McKibbin (1997).[6] The shocks chosen were to consumption, labour augmenting technical change, money demand and money supply, with each involving a 1% increase. These seem to correspond to what Lastrapes and Selgin envisage their shocks to be. The resulting $C(1)$ is

$$C(1)_{MSG2} = \begin{bmatrix} .19 & 0 & 0 & 0 \\ -.44 & .85 & 0 & 0 \\ .82 & .80 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Compared to the Lastrapes and Selgin solution, the MSG2 model provides all the elements of $C(1)$. Some of the elements in it clearly reflect the nature of MSG2; the final row for example arises because the first three shocks are "pure", in the sense that the money supply is not allowed to increase. If the MSG2 model had a money rule in it that responded to output and price changes we would expect to see some non-zero values in this row replacing those elements that are zero.

It is interesting to now ask what the consequences would be of adopting $C(1)_{MSG2}$ as the set of restrictions for determining $T$ from (5). First, because $C(1)_{MSG2}$ is non-singular, $T$ follows immediately from (5) as $A(1)C(1)_{MSG2}$. Second, the triangularity of $C(1)_{MSG2}$ suggests that it is a special case of Lastrapes and Selgin's formulation. However, there is a difference: when using the MSG2 information no assumption was made concerning the correlation between the $\varepsilon_t$ over any particular period of history, unlike the Lastrapes and Selgin set-up which maintains they are uncorrelated. In fact it would be inadvisable to impose orthogonal shocks in the "MSG2 case" unless that restriction was supported by the data. If it is felt to be desirable to impose some zero correlations between the shocks, it would be possible to use the MSG2 information selectively. For example, if only the first two columns of $C(1)_{MSG2}$ are assumed known one needs eight further restrictions to recover $\varepsilon_t$. Six of these are potentially provided by the assumption of zero correlations between the $\varepsilon_t$. Obviously, a lot hinges on whether the assumption that the economic shocks are uncorrelated is reasonable, a matter that has often been debated.

Using the data set corresponding to Lastrapes and Selgin - see Pagan and Robertson (1995) - we looked at the correlation between shocks that the MSG2 model would infer i.e. the knowledge of $C(1)$ and the estimated VAR is used to estimate $T$ and $\Sigma = T^{-1}\Omega(T')^{-1}$. The most striking was the high negative correlation between the IS and money demand shocks of $-0.95$. Money demand and money supply shocks had a much smaller correlation of 0.34. Thus, if one uses results from a model like MSG2 to determine shocks, it is not very likely that they can be taken as uncorrelated. Of course, this fact does not have any direct implications for the Lastrapes-Selgin shocks; since the VAR's of both models are identical one is just telling two different stories about the nature of the shocks that are needed to explain the data. It might be possible to discriminate between the models on other grounds than goodness of fit to the data; in the past this has meant using prior information about the signs and sizes of the impulse responses.

## 3.2 I(1) Variables are Cointegrated

Now let the $x_t$ be a vector of $n$ $I(1)$ series with $r$ cointegrating vectors, $0 < r < n$. In that case $x_t$ can be represented as the vector ECM (VECM)

$$A^*(L)\Delta x_t = \alpha\beta' x_{t-1} + e_t,$$

where $\alpha$ and $\beta$ are $n \times r$ matrices of rank $r$. Here $\beta$ is the co-integrating vector and $\beta' x_t$ are the $I(0)$ cointegrating relations. Engle and Granger (1987) show that $\Delta x_t$ can be represented as

$$\Delta x_t = D(L)e_t, \tag{7}$$

where $D(L) = I + D_1 L + ....$, and $D(1)$ is a singular matrix of rank $n - r$. They give the following expression for $D(1)$

$$D(1) = \beta_\perp(\alpha'_\perp A^*(1)\beta_\perp)^{-1}\alpha'_\perp,$$

where $\alpha_\perp$ and $\beta_\perp$ are $n \times (n-r)$ matrices of rank $(n-r)$ such that $\alpha'\alpha_\perp = 0$, $\beta'\beta_\perp = 0$, and $\alpha'_\perp A^*(1)\beta_\perp$ has full rank of $n - r$.

One difference between this case and that of a stationary system arises from the selection of estimates of $\left\{A_j^*\right\}$, $\{\alpha, \beta\}$ and $\Omega$ to summarize the data. To get these one needs to estimate a VECM, but what VECM? In the stationary case many theoretical models could be constructed to produce the same VAR, but the set of models that would produce identical VECM's is circumscribed by the need for them to have the same number of permanent shocks and to also exhibit the same values for $\beta$. As will be seen later this issue raises problems when it comes to imposing long-run restrictions from theoretical models. We note that both the choice of $n-r$ and the values ascribed to $\beta$ can be cast as testable restrictions on the VECM.

### 3.2.1 A Basic Permanent-Transitory Decomposition

Cointegration produces systems that are influenced by both permanent and transitory shocks. Stock and Watson (1988) showed that a co-integrated system would be driven by $n-r$ common trends and the shocks driving these common trends could be regarded as permanent. This leaves $r$ shocks to make up the full complement of $n$ and these are naturally defined as *transitory*. It is useful then to define the vector of permanent and transitory shocks as the $n \times 1$ vector $v_t = He_t$, where the $n \times n$ non-singular matrix $H$ is chosen so that the last $r$ elements in $v_t$ are transitory. With such a relation (7) will become

$$\begin{aligned} \Delta x_t &= D(L)H^{-1}He_t \\ &= F(L)v_t, \end{aligned} \tag{8}$$

where the covariance of $v_t$ is $V = H\Omega H'$.

From the definition of permanent and transitory shocks, it is clear that the last $r$ columns of

$$F(1) = D(1)H^{-1}.$$

must equal zero. This implies that $H$ must have the form

$$H = \begin{bmatrix} \alpha'_{\perp} \\ \rho' \end{bmatrix},$$

for any $n \times r$ matrix $\rho$ that makes $H$ invertible. To see why, observe that $H^{-1} = \begin{bmatrix} \rho_{\perp}(\alpha'_{\perp}\rho_{\perp})^{-1} & \alpha(\rho'\alpha)^{-1} \end{bmatrix}$ and $D(1)\alpha = 0$. Two choices that appear in the literature are Warne (1993) and Gonzalo and Granger (1995) who select $H$ as $H_W = \begin{bmatrix} \alpha'_{\perp} \\ \alpha' \end{bmatrix}$ and $H_G = \begin{bmatrix} \alpha'_{\perp} \\ \beta' \end{bmatrix}$ respectively.[7]

Which choice for $H$ should be used? When producing estimates of the common trends a standard approach, due to Stock and Watson (1988), is to decompose $D(1)$ as $\delta\gamma'$, where $\delta$ and $\gamma$ are $n \times (n-r)$ full rank matrices, and to then define the common trend as the random walk $\tau_t = \tau_{t-1} + \gamma' e_t$, with $x_t^p = \delta\tau_t$ giving the permanent components of the elements of $x_t$, formed by combining together the $n-r$ common trends. From the definition of $D(1)$, setting $\delta = \beta_\perp(\alpha_\perp' A(1)^* \beta_\perp)^{-1}$ and $\gamma = \alpha_\perp$ produces the requisite factorization. For any valid choice of $H$ the permanent shocks will be

$$v_t^p = \alpha_\perp' e_t,$$

while $\delta$ is equal to the first $n-r$ columns of $F(1)$. One potential difficulty that might decide between $H_W$ and $H_G$ is that, although $H_W$ must be non-singular, this is not so for $H_G$. Indeed, there may be instances in which $\beta = \alpha_\perp$. A simple example occurs in a two variable system when $\beta' = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\alpha' = \begin{bmatrix} \alpha_1 & \alpha_1 \end{bmatrix}$, i.e. the coefficients on the ECM terms are the same in both equations. Under this scenario $\alpha_\perp' = \begin{bmatrix} 1 & -1 \end{bmatrix} = \beta'$.[8]

To isolate a set of permanent shocks that are unique up to non-singular transformations, and some set of transitory shocks, one need go no further. Indeed, the only information needed to date would be knowledge of $r$, and the estimates from the VECM. However, the nature of the transitory shocks and their relation to the permanent shocks will vary according $H$. To isolate a unique set of transitory shocks, and to give the individual shocks distinct characteristics, one needs to incorporate some extra information about the system.

### 3.2.2 Informative Permanent-Transitory Decompositions

To incorporate any extra information we consider a new set of shocks $\varepsilon_t$ such that $v_t = T\varepsilon_t$. The non-singular $n \times n$ matrix $T$ is chosen so as to impose the desired characteristics for these shocks. To highlight the $\varepsilon_t$ we re-write (8) as

$$\begin{aligned} \Delta x_t &= F(L)TT^{-1}v_t \\ &= C(L)\varepsilon_t, \end{aligned}$$

where the covariance of $\varepsilon_t$ is $\Sigma = T^{-1}V(T^{-1})'$, and

$$C(1) = F(1)T \tag{9}$$

gives the long-run impacts of the $\varepsilon_t$ shocks upon the level of $x_t$. Now, assuming that the $\varepsilon_t$ can be regarded as containing $n-r$ permanent shocks, $C(1)$, like $F(1)$, must also have its last $r$ columns equal to zero. Such a requirement also constrains the nature of $T$ and, to see how, partition $C(1)$ and $F(1)$ conformably with the permanent and transitory shocks to get

$$\begin{bmatrix} C_{11}(1) & 0 \\ C_{21}(1) & 0 \end{bmatrix} = \begin{bmatrix} F_{11}(1) & 0 \\ F_{21}(1) & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{11}$ is $(n-r)\times(n-r), T_{21}$ is $r\times(n-r)$ etc. Using the notation that $F_p(L)$ collects together the first $n-r$ columns of $F(L)$, and $F_s(L)$ collects the last $r$ columns, we have $F_s(1) = 0$ and $F_p(1)T_{12} = 0$, implying that $T_{12} = 0$. Thus re-combination of the permanent shocks in $v_t$ is the only permissible operation when defining the permanent shocks in $\varepsilon_t$, and the latter are therefore $\varepsilon_t^p = T_{11}^{-1}v_t^p$, for any non-singular $T_{11}$. The covariance of $\varepsilon_t^p$, $\Sigma_{11}$, can then be solved from the expression

$$T_{11}\Sigma_{11}T_{11}' = V_{11}, \tag{10}$$

where $V_{11}$ is the covariance of $v_t^p$.

To produce permanent shocks having specific characteristics some extra information must be used to guide the choice of $T_{11}$. This information may come in the form of knowledge about the long run multiplier matrix $C_p(1)$. There are three cases to examine, depending on whether all, some, or none of $C_p(1)$ is assumed known.

**(a) Finding Permanent Shocks, all of $C_p(1)$ known.** To utilize such knowledge concatenate (9) and the fact that $T_{12} = 0$ to deduce

$$C_p(1) = F_p(1)T_{11}. \tag{11}$$

Because $F_p(1)$ has full column rank one can then solve for $T_{11}$ as

$$T_{11} = (F_p(1)'F_p(1))^{-1}F_p(1)'C_p(1). \tag{12}$$

Since $C(1) = \left[\begin{array}{cc} F_p(1)T_{11} & 0 \end{array}\right]$, the implied trend component of $x_t$ can then be expressed as $x_t^p = C_p(1)\phi_t$, with $\phi_t = \phi_{t-1} + \varepsilon_t^p$.

The dynamic response of $\Delta x_t$ to changes in $\varepsilon_t^p$ can be obtained by decomposing (8) into the separate effects of permanent and transitory elements to give

$$\Delta x_t = F_p(L)v_t^p + F_s(L)v_t^s \tag{13}$$

$$= F_p(L)T_{11}\varepsilon_t^p + F_s(L)(T_{21}\varepsilon_t^p + T_{22}\varepsilon_t^s). \tag{14}$$

Evidently there are two channels through which $\varepsilon_t^p$ can influence $\Delta x_t$; a direct one through $F_p(L)T_{11}$, and an indirect one which comes via the fact that, in the data, $v_t^s$ may be correlated with $\varepsilon_t^p$. The obvious way to capture this indirect effect is by regressing $v_t^s$ on $\varepsilon_t^p$. The net contribution of the permanent shocks is then captured by the dynamic multipliers

$$F_p(L)T_{11} + F_s(L)\theta, \tag{15}$$

where $\theta = \text{cov}(v_t^s, \varepsilon_t^p)\Sigma_{11}^{-1} = T_{21} + T_{22}\Sigma_{21}\Sigma_{11}^{-1}$ is the regression coefficient, and $\Sigma_{21}$ is the covariance between $\varepsilon_t^s$ and $\varepsilon_t^p$. This choice for $\theta$ ensures that the linear combination $v_t^s - \theta\varepsilon_t^p$ is uncorrelated with $v_t^s$.[9] Notice that the quantity being computed in (15) gives the impact of changes in $\varepsilon_t^p$ upon $\Delta x_t$ regardless of the assumed correlation between $\varepsilon_t^p$ and $\varepsilon_t^s$. One might wish to consider the impacts of changing $\varepsilon_t^p$ while keeping $\varepsilon_t^s$ constant, i.e. setting $\Sigma_{21} = 0$. This has the immediate implication that $\theta$ is equal to $T_{21}$. But zero correlation itself has no implications for the computation of the dynamic responses, since $\theta$ is always obtained via the regression of $v_t^s$ on $\varepsilon_t^p$.

It is also important to realize that $F_p(1)$ in (12) has to be derived from a VECM which sets $\beta$ to the vectors that are compatible with $\beta' C(1) = 0$. Many estimators of $\beta$ exist that do not use any information regarding the form of $C(1)$ other than the number of permanent shocks. For example, if one uses Johansen's (1988) estimator to produce a $\tilde{\beta}$ there is no guarantee that $\tilde{\beta}' C(1) = 0$. If $\tilde{\beta}' C(1) \neq 0$ one cannot use the VECM that underlies Johansen's approach to construct the quantities in (12). Of course it is possible to test if a known $\beta$ is compatible with the estimate $\tilde{\beta}$. But, if these $r(n-r)$ restrictions are rejected, one is left to ponder the validity of the choice of $C(1)$ used.

**(b) Finding Permanent Shocks, some of $C_p(1)$ known.** The most prominent paper dealing with this case is King et al (1991) (KPSW). Several papers have since appeared using their technology; a short listing being Mellander et al (1992), Fisher, Fackler and Orden (1995), and Fisher (1996). KPSW assume that $C_p(1) = S\Lambda$, where $S$ is a known full rank $n \times (n-r)$ matrix with the property that $\beta' S = 0$, and $\Lambda$ is some full rank $(n-r) \times (n-r)$ lower triangular matrix. From expression (12) we have that $T_{11} = (F_p(1)' F_p(1))^{-1} F_p(1)' S\Lambda = R\Lambda$, say, showing that the $(n-r)^2$ unknown elements in $T_{11}$ have been reduced to the $(n-r)(n-r+1)/2$ unknown elements in $\Lambda$. This is exactly the number of independent elements in $V_{11}$. Assuming additionally that $\Sigma_{11} = I_{n-r}$, and making the substitution of $T_{11} = R\Lambda$ into equation (10), shows that $\Lambda$ can be found by performing a Cholesky decomposition upon $R^{-1} V_{11} (R')^{-1}$. Once $T_{11}$ is determined the multipliers can be computed from (15). KPSW interpret these multipliers as measuring the effect of changes in $\varepsilon_t^p$ when the transitory shocks are held fixed, and this is achieved by assuming $\Sigma_{21} = 0$. As we noted, this has no implications for the computation of the multipliers, but does allow them to isolate $T_{21}$.

One important qualification needs to be made about the KPSW procedure. Some action has to be taken to ensure that $\beta' C_p(1) = 0$. If one sets $\beta$ to $\tilde{\beta}$, say Johansen's estimator, this would mean that $\tilde{\beta}' S = 0$. This constraint is enforced by KPSW in the way that $S$ is chosen. Thus $C(1)$ is not really specified independently of the data but is constructed, in part, from it. In particular, $S$ is only known after a VECM is estimated, and hence the constraint $\tilde{\beta}' S = 0$ is specific to the estimator of $\beta$ employed. Whether such "reverse engineering" produces an estimated $C(1)$ which makes economic sense, particularly given the triangular assumption for $\Lambda$, is problematic. KPSW tell a story about this in their paper but it is frequently a lot harder to do in other contexts. Moreover, as we will note later in the context of their empirical work, the story can get rather muddied, since it needs to be about $C(1)$ and that is a product of two items, $S$ and $\Lambda$, and their interaction makes interpretation much more difficult. Certainly, it does not represent a solution that is easy to generalize.

**(c) Finding Permanent Shocks, none of $C_p(1)$ known.** If no elements of $C_p(1)$ are known the extra information to determine $T_{11}$ must come from some other source. For example, restrictions might be imposed directly upon the structure of $T_{11}$ as well as that of $\Sigma_{11}$. When $T_{11}$ is made triangular and $\Sigma_{11} = I_{n-r}$, (10) shows that $T_{11}$ can be found by performing a Cholesky decomposition on $V_{11}$. This method is proposed in Gonzalo and Ng (1996), and is described in a more general context by Yang (1998).[10] One advantage of this approach is that it can be done with standard packages; a disadvantage may be that it uses little economic information.

**(d) Finding Transitory Shocks** In the discussion above the emphasis has been upon the isolation of permanent shocks. In order to identify the transitory shocks $\varepsilon_t^s = T_{22}^{-1}(v_t^s - T_{21}\varepsilon_t^p)$ some further assumptions are required. From the covariance relations $T\Sigma T' = V$, it follows that making the transitory and permanent shocks uncorrelated, i.e., setting $\Sigma_{21} = 0$, produces

$$T_{21}\Sigma_{11}T'_{11} = V_{21}. \tag{16}$$

Once $T_{11}$ and $\Sigma_{11}$ are known it is clear that $T_{21}$ can be determined from (16) as $T_{21} = V_{21}\Sigma_{11}^{-1}(T'_{11})^{-1}$.[11] With this information the set of transitory "economic" shocks are identified up to a normalization factor for any non-singular $T_{22}$. To find $T_{22}$ turn to the last of the equations available from partitioning $T\Sigma T' = V$, namely

$$T_{21}\Sigma_{11}T'_{21} + T_{22}\Sigma_{22}T'_{22} = V_{22}. \tag{17}$$

Combining an assumption that the covariance of the transitory shocks, $\Sigma_{22}$, equals $I_r$ with the symmetry of $V_{22}$, (17) points to the need for an additional $r(r-1)/2$ restrictions. When $T_{22}$ is made triangular, the unknown elements can be found by performing a Cholesky decomposition of $V_{22} - T_{21}\Sigma_{11}T'_{21}$, as in Gonzalo and Ng (1996), but it is again unclear what the economic meaning of such a constraint is.

An alternative approach is to try to exploit information about $C(L)$, specifically the impact multipliers $C_0$. Defining $F_0 = H^{-1}$, and partitioning it conformably with $\varepsilon_t^p$ and $\varepsilon_t^s$, allows us to write the impact responses to the permanent and transitory shocks as

$$C_0 = F_0 T = \begin{bmatrix} F_{p,0}T_{11} + F_{s,0}T_{21} & F_{s,0}T_{22} \end{bmatrix}.$$

In general, a set of restrictions on $T_{22}$ do not translate into the same restrictions on the impact responses $C_{s,0} = F_{s,0}T_{22}$, because the elements of $F_{s,0}$ are involved. This is clear from the fact that $T_{22} = (F'_{s,0}F_{s,0})^{-1}F'_{s,0}C_{s,0}$. Similarly, imposing restrictions directly upon $C_{s,0}$ is complicated by the presence of $F_{s,0}$. Thus, for example, restricting $C_{s,0}$ to be lower triangular need not correspond to the same restriction on $T_{22}$. This point has been made before by Englund, Vredin and Warne (1994), and the complexities it raises probably accounts for the fact that few studies have attempted to isolate the impact of the transitory shocks through information about $C(L)$.

So far the emphasis in this section has been upon obtaining shocks that have specific implications for the long-run properties of impulse response functions. But, as section 2 detailed, there is another history in which the shocks are treated as the error terms of structural equations. The impulse responses are then determined by estimating these structural relations and solving for the MA representation. KPSW did use the language of structural and reduced form shocks but did not formally write out any structural relations to be estimated. Most of those following in their footsteps have also adopted the language but have been suitably vague about what the structure was.[12] As is evident from the above derivations there is really no specific connection with a structural relation in the KPSW treatment.

If one wanted to proceed in the latter direction how would one adapt the solution in the stationary case to handle cointegrated variables? One simple solution is always available. The VECM is a VAR that imposes the cointegrating restrictions and so its errors remain as $e_t = B_0^{-1}\varepsilon_t$. Therefore, the determination of $B_0$ can be done through the usual process of exclusion restrictions on $B_0$ and $\Sigma$, except that the VAR parameters are now obtained from the VECM. There is no necessary connection with permanent or transitory shocks in this solution however, and it is likely that all shocks will have permanent components.

Suppose instead that the shocks associated with structural equations can be partitioned into permanent and transitory. Specifically, take the first $n - r$ structural equation shocks as being permanent and the remaining $r$ as transitory. Then, after performing the same sequence of transformations as previously, one obtains the representation

$$\Delta x_t = F(L)HB_0^{-1}\varepsilon_t.$$

In order for this to be written as $\Delta x_t = F(L)T\varepsilon_t$ it must be the case that $HB_0^{-1} = T$ and so $T$ becomes a non-singular transformation of $B_0^{-1}$. Because $H$ is also involved, there seems no general way to ensure that a particular set of restrictions on $B_0$ will give a $T$ that preserves the requisite lower block triangular form. However, the case when $B_0$ and $H$ are both lower triangular has received some attention in the literature

- see for example, Cochrane (1994), Gonzalo and Ng (1996), and Ribba (1997). In that case an assumption that $B_0$ is lower triangular ensures that $T$ is lower triangular. For $n = 2$, the necessary exclusion restriction on $H$ is easily testable from the VECM since it amounts to excluding the error correction term from the first equation so as to eliminate any levels effect.

## 3.3 Deriving the Impulse Responses

In most instances the objective of the analysis is to find impulse responses with respect to shocks. This task can be performed in two steps. In the first, $D(L)$ is computed i.e. the impulse responses of $\Delta x_t$ to the errors $e_t$. In the second, these are recombined as $D(L)TH^{-1}$ in order to produce the responses to the permanent and transitory shocks. When $x_t$ are either stationary or not cointegrated, $D(L)$ is easily found by inverting $A(L)$, but, with cointegration, $A(L)$ can not be directly inverted. There have been a number of methods advanced to solve this problem. A useful one is that in Mellander Vredin and Warne (1992), and it is adopted here.

Define $M$ as an $n \times n$ non-singular matrix $\begin{bmatrix} \Gamma \\ \beta' \end{bmatrix}$, where $\Gamma$ is an $(n-r) \times n$ selection matrix, $\Phi_{\perp}(L) = \begin{bmatrix} (1-L)I_{n-r} & 0 \\ 0 & I_r \end{bmatrix}$ and $\Phi(L) = \begin{bmatrix} I_{n-r} & 0 \\ 0 & (1-L)I_r \end{bmatrix}$.[13] Then set $w_t = \Phi_{\perp}(L)Mx_t$, where the variables $w_t$ will be $I(0)$, since the first $(n-r)$ components represent linear combinations of $\Delta x_t$, and the last $r$ constitute the cointegrating errors $\beta' x_t$. This representation is a generalization of Campbell and Shiller (1988). Because $w_t$ is $I(0)$ it will be represented as a stationary VAR of the form

$$E(L)w_t = Me_t, \tag{18}$$

and will have an associated MA representation

$$w_t = E^{-1}(L)Me_t = E^{-1}(L)MH^{-1}v_t$$

$$
\begin{aligned}
&= E^{-1}(L)MH^{-1}T\varepsilon_t \\
&= Q(L)\varepsilon_t.
\end{aligned}
$$

The following relationships between $E(L), A(L)$ and $D(L)$ can then be established:

$$
E(L) = M[A^*(L)M^{-1}\Phi(L) + \gamma^* L]
$$

$$
D(L) = M^{-1}\Phi(L)E(L)^{-1}M,
$$

where $\gamma^* = [\ 0 \quad \alpha\ ]$. Using these formulae $Q(L)$ can be constructed from $M$, $A^*(L)$ and $T$.

# 4 Mixtures of I(1) and I(0) Variables

Cases arise when there are a mixture of $I(1)$ and $I(0)$ variables within a system. An example would be Gali (1992) in which there are two $I(1)$ variables, the log of output and the nominal interest rate, and two $I(0)$ variables, the real interest rate and the rate of growth of real money balances. One way to handle this complication is to act as if all variables are $I(1)$ but with the $I(0)$ variables cointegrating with themselves. Thus any cointegrating vectors among the $I(1)$ variables are augmented with the artificial vectors effecting this. In terms of Gali's model let the $I(0)$ variables be the third and fourth, in which case the artificial vectors are $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. With this re-definition it would be possible to do the analysis as described in the section on co-integration. However, in doing so one may have lost some information. To see this most clearly assume that the first $n_1$variables in $x_t$ are $I(1)$, the last $n_2$ are $I(0)$, so that $n = n_1 + n_2$, and there is no cointegration among the $I(1)$ variables. Designate the $I(1)$ variables by $x_{1t}$ and the $I(0)$ by $x_{2t}$. In the section on co-integration the analysis worked with a MA representation of $\Delta x_t$, but it is more natural to define one in terms of $\Delta x_{1t}$ and $x_{2t}$, viz.

$$\begin{pmatrix} \Delta x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} C_{11}(L) & C_{12}(L) \\ C_{21}(L) & C_{22}(L) \end{pmatrix} \begin{pmatrix} \varepsilon_t^p \\ \varepsilon_t^s \end{pmatrix} \tag{19}$$

Now if there is some knowledge about $C_{21}(L)$, i.e. the impulse responses showing the effect of permanent shocks upon the $I(0)$ variables, it should presumably be used. Working with the representation above allows that to happen. If, instead, one proceeded as in the cointegrated case, the system will involve $\Delta x_{2t}$ rather than $x_{2t}$. This has the effect of preserving only the information contained in $C_{11}(1)$, since $\Delta x_{2t} = (1 - L)C_{21}(L)\varepsilon_t^p$ and, when $L = 1$, any information about $C_{21}(1)$ would disappear from such a system. Values for $C_{21}(1)$ may well be available from calibrated models. For example, in Gali's system, it seems reasonable to treat one of the permanent shocks of the system as being to the money growth rate (given that he assumes that the growth rate in nominal money is $I(1)$). Then, simulations of a permanent shock to money growth in MSG2 allows one to calculate the effect of such a shock upon the real interest rate at different time horizons. Summing the latter gives a value of $-1.6$, and hence one of the elements of $C_{21}(1)$. It might be noted that such information is also available about the impact of a permanent shock upon cointegrating errors, since these would just be $\beta'\Psi(L)$, and one might wish to use it as well when extracting estimates of shocks. In the presentation of section 3.2 only values of $\Psi_\infty$ were used. Obviously the appropriate way to incorporate this extra information would be to use the system given as (18). Another example of this framework is the two variable system of Blanchard and Quah (1989), where $n_1 = 1$. Their restriction is that $C(1)$ in (19) is triangular with the shocks $\varepsilon_t = T^{-1}e_t$ being uncorrelated, and where $e_t$ are the errors in a VAR involving $\Delta x_{1t}$ and $x_{2t}$. Crowder (1995) observes that this model can be represented using cointegration language by making $\beta = (0, 1)'$.

# 5 An Application

In this section we provide a small empirical illustration of the differences amongst the various techniques described above. The quarterly data are the same as used in KPSW and span the period 1954:Q1 to 1988:Q4 for six variables - the logs of private sector output $(o_t)$, consumption $(c_t)$, investment $(i_t)$, and the real money stock $(m - p_t)$, together with the nominal interest rate $(r_t)$ and the annualized rate of inflation $(dp_t)$. All variables are treated as being $I(1)$ with $r = 3$, and the fitted VAR includes a constant and 6 lags.[14]

Since one is dealing with a theoretical model it is necessary to decide on what the three permanent shocks to the system are. From their discussion, labour augmenting technical change and money growth are suitable candidates; the latter because of their treatment of the inflation rate as being $I(1)$. The third is more difficult to decide upon. KPSW treat it as a source of permanent shifts in the real interest rate and there are many candidates for that. One that has been important over the sample period would be oil price shocks, so that we use this for illustrative purposes. The effects of permanent increases in these shocks upon the six nominated variables were then simulated in the MSG2 model; the magnitude of the shocks were one percentage point increases for the first two and a 100% rise in oil prices for the last. Below we give the first three columns of $C(1)$ coming from such a simulation - the remaining three are all zero. Variables and shocks are arranged in the order described above. Thus the first column shows the effects of an unanticipated one percentage point increase in the rate of labour augmenting technical change upon the six variables, with the third row of this column giving its impact upon the level of investment. For flow variables these impacts are percent changes, whereas for inflation and the interest rate they would need to be multiplied by a factor of one hundred to produce annualized basis point increases.

$$C_p(1)_{MSG2} = \begin{bmatrix} 0.82 & 0 & -1.16 \\ 0.85 & 0 & -4.12 \\ 1 & 0 & -5.33 \\ 0.79 & 4 & -1.50 \\ 0 & 1 & 0.32 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

Perhaps the result that is most curious is that a rise in the money growth rate leads to a rise in real balances in the long run. The reason for this is that prices are measured by the GDP deflator in KPSW's work. If, instead, money had been deflated by the CPI this effect would be absent as the CPI rises virtually one for one with the money supply in MSG2. Also, notice that a productivity shock does not raise consumption, output and investment by the same amount in the long run because of the open economy aspects of MSG2 - see Mellander et al (1992) for a discussion of this issue.

It is interesting to compare $C_p(1)_{MSG2}$ with that found using the KPSW approach. KPSW have $C_p(1)_{KPSW} = S\Lambda$ and construct the $S$ matrix from the estimated $\beta$ reported in their Table 2 (p. 828). We have normalised on the diagonal of $\Lambda$ so that $C_p(1)_{KPSW}$ gives the long-run responses to 1 unit shocks.

$$C_p(1)_{KPSW} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & .0033 \\ 1 & 0 & -.0028 \\ 1.197 & -.0134 & -.0134 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \Lambda_{21} & 1 & 0 \\ \Lambda_{31} & \Lambda_{32} & 1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ .947 & -.002 & .0033 \\ 1.045 & .002 & -.0028 \\ 1.357 & -.004 & -.0134 \\ -11.891 & .294 & 1 \\ 4.033 & 1 & 0 \end{bmatrix}$$

The interpretation of the shocks defined in KPSW is given in terms of the matrix $S$ (p.831). But, as can be seen from (21), the long-run responses are measured by $C_p(1)_{KPSW}$ and not $S$; the two will differ whenever $n-r > 1$ and the permanent shocks are constructed so as to be uncorrelated. Without the latter requirement one could proceed by setting $C(1) = \begin{bmatrix} S & 0 \end{bmatrix}$. Consequently, the story told by KPSW about the nature of their permanent shocks, whilst consistent with $S$, does not actually hold in their empirical model, as it is not consistent with $C_p(1)_{KPSW}$.

Because $C_p(1)_{MSG2}$ is taken as known it is possible to estimate what the correlation between the "MSG2 type" permanent shocks would need to be in order to replicate the data. The highest correlation amongst the permanent shocks is 0.51 between the productivity and oil price shocks, while the money growth shock is almost orthogonal to the oil price shock and negatively correlated with the productivity shock. Specification of $C_p(1)_{MSG2}$ and $C_p(1)_{KPSW}$ also imply certain cointegration properties. To examine these we proceed by partitioning $C(1)$ as before to give $\beta' C_p(1) = 0$, resulting in $r \times (n-r)$ equations in $n \times r$ unknowns. Accordingly, $r^2$ elements in $\beta$ have to be prescribed in order to find a unique set of cointegrating vectors. This is the same condition as derived by Pesaran and Shin (1997) through a different route, and is the foundation of the "structural cointegration" literature. A simple solution is to adopt some normalization, for example $\beta' = \begin{bmatrix} -I_r & \beta_2' \end{bmatrix}$, as that enables one to solve for $\beta_2'$ as $C_{11}(1)C_{21}(1)^{-1}$. Assuming that the normalization is upon consumption, investment and real money balances, the cointegrating vectors underlying the MSG2 model are then found to be

$$\begin{aligned} c_t &= 1.04o_t - .09(r_t - dp_t) \\ i_t &= 1.22o_t - .12(r_t - dp_t) \\ m_t - p_t &= .96o_t - .012r_t + .052dp_t, \end{aligned} \tag{22}$$

which can be contrasted with those used by KPSW

$$\begin{aligned} c_t &= o_t + .0033(r_t - dp_t) \\ i_t &= o_t - .0028(r_t - dp_t) \\ m_t - p_t &= 1.197o_t - .013r_t. \end{aligned} \tag{23}$$

Unlike the MSG2 formulation KPSW impose unit output coefficients and work with smaller real interest rate effects in the consumption and investment relations. Also, KPSW have a larger output coefficient in the relation for real money balances, while the inflation coefficient is set to zero.[15]

The first columns of $C_p(1)_{MSG2}$ and $C_p(1)_{KPSW}$ give the long run responses to a positive permanent shock to the level of productivity in the two models. Figure 1 plots the set of impulse responses to this shock, as computed from (15). Probably the main differences lie in the behaviour of the nominal interest rate and the inflation rate. Using the KPSW story interest rates would be expected to increase after a favourable productivity shock. This seems unusual but might be explained by the fact that the VAR equations will tend to represent interest rates over this historical period with a "Taylor rule", whereby a rise in output would induce a rise in nominal interest rates. Of course the rise in output in this case should be disregarded by the monetary authorities, as it comes from a positive supply side response rather than from demand.

The second column of $C_p(1)_{MSG2}$ and $C_p(1)_{KPSW}$ give the long run responses to a permanent shift in the rate of money growth. Figure 2 plots the set of impulse responses to this shock. Again, there are marked differences in interest rate responses from each model, although this time it is in magnitudes rather than in signs. Since money growth has risen by 1% per annum the MSG2 story has inflation and the nominal interest rate both rising eventually by 100 basis points. Under the KPSW story however, although the inflation rate rises by that amount, the nominal interest rate does not, i.e. a permanent shift in the money growth rate in their model actually permanently lowers the real interest rate.

Figure 3 plots the log output series together with the trend components implied by (21) and (20). There are some differences between the two trend components but the most striking features relate to the recessions. One can decompose the permanent part of a series into the contributions from each type of shock and, for both KPSW and MSG2, the permanent part of output turns out to be overwhelmingly due to productivity. This feature suggests that neither MSG2 nor KPSW see productivity variations as being the "cause" of recessions, although Figure 3 does indicate that the MSG2 story would be compatible with it having some role in the mid 1970's recession.

Figure 1: Responses to Productivity Shock: MSG2 (—) and KPSW (- -)

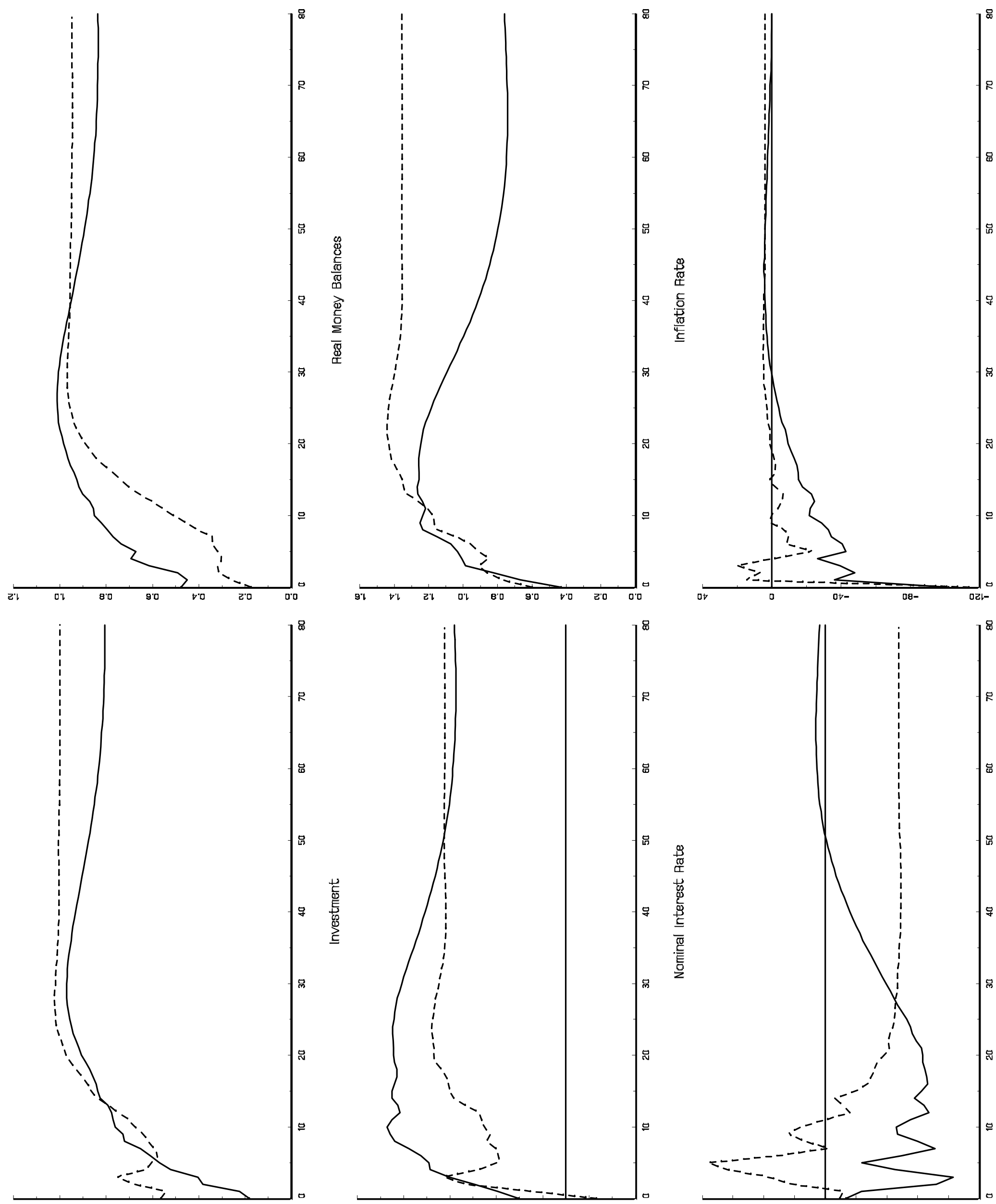

Figure 2: Responses to Money Growth Shock: MSG2 (—) and KPSW (- -)

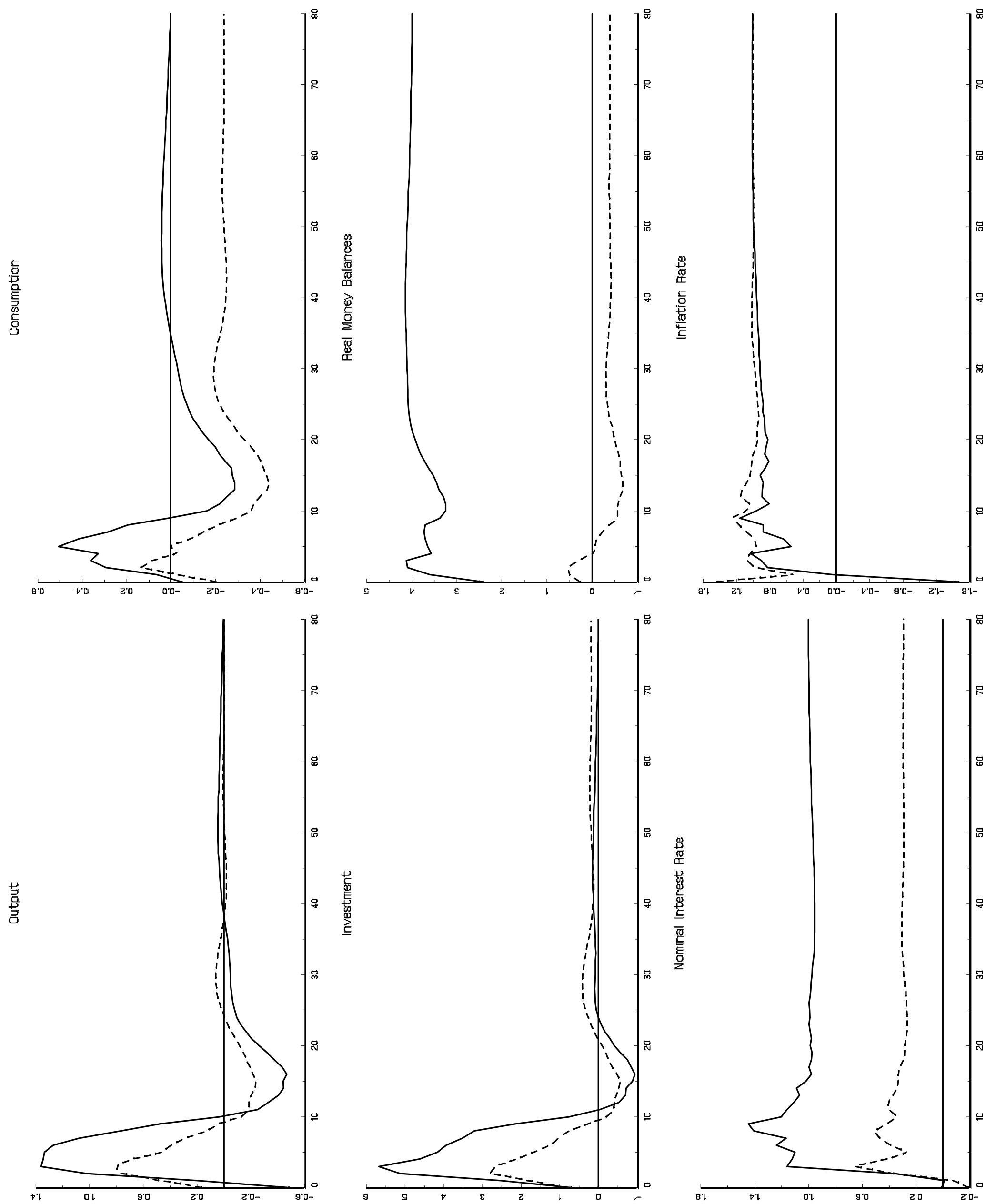

Figure 3: Estimated Trend Component of log Output


# 6 Conclusion

The paper has tried to present a systematic approach to the literature dealing with the decomposition of multivariate time series into their permanent and transitory components. Because the shocks underlying such components frequently occur in the construction of theoretical models we have tried to emphasise how the information from such models might be brought to bear upon the task of separation. The central thrust of our paper was that theoretical models can be regarded as producing information about the magnitude of impulse responses, particularly the size of cumulated responses. This information is rarely fully exploited in much of the existing literature, which has tended to focus upon a subset of the implications, most notably those relating to co-integration.

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# Notes

[1]We have excluded discussion of the role of intercepts, deterministic trends and initial conditions here and in what follows. For I(1) series these are important factors that can influence the testing for and estimation of cointegrating relations. But, conditional on their specification, they are of little consequence for impulse response analysis. A discussion of the role deterministic terms and exogenous variables in VAR modelling is provided by Pesaran and Smith (1998).

[2]With $A_j$ and $\Omega$ *fixed at some estimated values* the estimates of the "structural" coefficients are obtained as the solution to the implied systems of equations, and the solution will vary with the specific identifying restrictions used.

[3]Of course, this order condition is not sufficient by itself to ensure the shocks are uniquely determined.

[4]Although we use the term "calibrated model" somewhat loosely it would be expected that such a model has the dual characteristics of being based on a coherent theoretical framework and of providing some quantitative measures of the impacts of shocks.

[5]In a stationary system $C(1)$ is simply a summary of information about the lag distribution and does not say anything about the level of $y_t$ in the "long-run" *per se*.

[6]The MSG2 model is a multi-country dynamic intertemporal general equilibrium model of the world economy. It has a well determined long run being driven by a Solow-Swan-Ramsey neoclassical growth model, with exogenous technical progress and population growth. In the short run, however, the dynamics of the global economy towards this growth path are determined by a number of Keynesian style rigidities in the goods and labor markets. Households and firms are assumed to maximize intertemporal utility and profit functions subject to intertemporal budget constraints. In the short run some proportion of firms and households use optimal rules of thumb rather than recalculating the entire intertemporal equilibrium of the model. Wages

are assumed to adjust slowly to clear labor markets subject to the institutional characteristics of labor markets in different economies. Intertemporal budget constraints are imposed so that all outstanding stocks of assets must be ultimately serviced, and asset markets are efficient, in the sense that asset prices are determined by a combination of intertemporal arbitrage conditions and rational expectations.

[7]A choice suggested by Kasa (1992) has $H_K = \begin{bmatrix} \beta_\perp, & \beta \end{bmatrix}'$. This is unsatisfactory since $H_K^{-1} = \begin{bmatrix} \beta_\perp(\beta_\perp'\beta_\perp)^{-1} & \beta(\beta'\beta)^{-1} \end{bmatrix}$ and $D(1)\beta \neq 0$.

[8]Trevor Breusch has pointed out that this case could arise if the variables were from a system of demand relations subject to an adding up condition. This restriction normally forces the serial correlation in the equations to be of the same type - see Berndt and Savin (1975).

[9]When doing policy analysis one could make a case for simply using $F_p(L)T_{11}$ as the impulse responses to a permanent shock as that assumes that the correlation between $\varepsilon_t^p$ and $v_t^s$ is zero, which may be a better assumption in the policy period. McKibbin et al (1998) use this particular variant.

[10]We saw Yang's paper just as the final re-write of this paper was taking place. It follows a similar approach to ours. For instance, his $N$ matrix is our $T_{11}^{-1}$.

[11]Recall, $T_{21}$ is the coefficient matrix in a regression of $v_t^s$ on $\varepsilon_t^p$ when $\Sigma_{21} = 0$.

[12]Although Englund at al (1994, p 144) say "By structural shocks we mean innovations which- in contrast to the VAR residuals- are independent".

[13]They indicate that $\Gamma$ can always be set to $\beta_\perp'$. However, if $\beta$ was known, $\Gamma = \begin{bmatrix} I_{n-r} & 0 \end{bmatrix}$ would be available since $\beta'$ can then always be re-written in the form $\beta' = \begin{bmatrix} \beta_1' & I_r \end{bmatrix}$ by normalization.

[14]A description of the construction of the data and the data's summary time series properties is presented in KPSW. The data set and various programs necessary to replicate their results are avaliable at Mark Watson's Princeton Univeristy home page: http://www.wws.princeton.edu/˜mwatson/.

[15]The $\hat{\beta}$ used in KPSW and reported in (23) is not actually estimated by Johansen's maximum likelihood method. In particular, KPSW use an estimator that explicitly

utilizes the normalization of $\beta$ as $\begin{bmatrix} -I_r & \beta_2' \end{bmatrix}'$. The LR statistic for a test of the $r(n-r) = 9$ restrictions on the VECM implied by the choice of $\beta$ used in (23) is 20.49 (p-value $=$ 0.02). An LR test of the 9 restrictions implied by the choice of $\beta$ in (22) gives a value of 51.67 (p-value $=$ 0.00). Thus, it seems that both the KPSW and MSG2 cointegrating vectors are rejected in the context of a VECM representation estimated by Johansen's method. This conclusion is invariant to the lag length used.